Fully indecomposable non-convertible (0,1)-matrices

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Notion of convertibility

Definition
Let $A = (a_{ij})$ be a square matrix of order $n$ and $S_n$ is a symmetric group on $n$ elements, then

$$
\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}
$$

$$
\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}
$$

Definition
Matrix $A \in M_n(0, 1)$ is convertible if there is matrix $X \in M_n(\pm 1)$ such that $\text{per}(A) = \det(A \circ X)$, where $\circ$ is Hadamard multiplication of matrices.
Examples of convertibility

Example

\[
\text{per} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \det(\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix})
\]

Example

Matrix \( J_3 \) is non-convertible.

Example

- Number of domino tiling’s
- Number of derangements of order \( n \)
- Ménage numbers
- Number of perfect matching in bipartite graph
Gibson bounds

Definition
Let $\Omega_n \in \mathbb{Z}_n$. We say that $\Omega_n$ is upper Gibson bound of convertibility for matrices of order $n$ if every matrix $A \in M_n(0,1)$ with $\text{per}(A) > 0$ and more than $\Omega_n$ positive elements is non-convertible.

Definition
Let $\omega_n \in \mathbb{Z}_n$. We say that $\omega_n$ is lower Gibson bound of convertibility for matrices of order $n$ if every matrix $A \in M_n(0,1)$ with less than $\omega_n$ positive elements is convertible.
Theorem (Gibson)

Let \( A \in M_n(0,1) \), \( n \geq 3 \) and \( \text{per}(A) > 0 \). If matrix \( A \) is convertible then \( \nu(A) \leq \frac{n^2 + 3n - 2}{2} = \Omega_n \). If \( \nu(A) = \Omega_n \) and \( A \) is convertible then \( A \) is permutationally equivalent to \( G_n \), where

\[
G_n = \begin{pmatrix}
1 & 1 & 0 & \ldots & 0 \\
1 & 1 & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & 1 & \ldots & 1 \\
1 & 1 & 1 & \ldots & 1
\end{pmatrix}
\]

Theorem (Little || Dolinar, Guterman, Kuzma)

Let \( A \in M_n(0,1) \) and \( n \geq 3 \). If \( \nu(A) < \omega_n = n + 6 \) then \( A \) is convertible.
Fully indecomposable matrices

Definition

Matrix $A$ is partially decomposable if there are permutation matrices $P, Q$ such that $PAQ$ is upper block triangle matrix.

Definition

If matrix $A$ is not partially decomposable then it is fully indecomposable.

Example

Let

$$A = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}$$

The matrix $A$ is partially decomposable and the matrix $B$ is fully indecomposable.
Convolution

Definition
Let $A \in M_n(0,1)$ and let the 1-st row of $A$ has exactly two non-zero entries $a_{11}, a_{12}$. Then the convolution of $A$ in the 1-st row is the following matrix $S_1(A) \in M_{n-1}$,

$$S_1(A) = \begin{pmatrix}
\max(a_{21}, a_{22}) & a_{23} & \ldots & a_{2n} \\
\max(a_{31}, a_{32}) & a_{33} & \ldots & a_{3n} \\
\ldots & \ldots & \ldots & \ldots \\
\max(a_{n1}, a_{n2}) & a_{n3} & \ldots & a_{nn}
\end{pmatrix}$$

Theorem
Let $A \in M_n(0,1)$. Let the first row of $A$ have exactly two non-zero entries: $a_{11}, a_{12}$, and let $S_1(A)$ be the convolution of $A$. Then $A$ is convertible if and only if $S_1(A)$ is convertible.
Lower bound for fully indecomposable matrix

**Theorem**

Let $A \in M_n(0, 1)$ and $\mu(A)$ be its vector of row sums. Assume that all entries of $\mu(A)$ except perhaps at most two are less than or equal to 2. Then $A$ is convertible.

**Theorem**

Let $A \in M_n(0, 1)$ be fully indecomposable. If $\nu(A) \leq 2n + 2$ then $A$ is convertible.
Exactness of the bound

Example

Matrix

\[
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 1 & 1 \\
0 & 0 & 0 & \ldots & 1 & 1 & 1 \\
1 & 0 & 0 & \ldots & 0 & 1 & 1 \\
\end{pmatrix}
\]

has $2n + 3$ ones, fully indecomposable and non-convertible.
Symmetric matrices with zero diagonal

**Theorem**
Suppose a symmetric matrix $A = (a_{ij}) \in M_n(0,1)$ with the zero diagonal has at most $2n + 4$ nonzero entries, and has in each row at least 2 nonzero entries. Then $A$ is convertible.

**Theorem**
Suppose a symmetric fully indecomposable matrix $A = (a_{ij}) \in M_n(0,1)$ with the zero diagonal has at most $2n + 4$ nonzero entries. Then $A$ is convertible.
Exactness of the bound for symmetric matrices

**Theorem**

Let \( n \geq 5 \) be an odd integer, \( J = \sum_{i=1}^{n-1} E_{i,i+1} \) be an upper-triangular Jordan nilpotent and consider a symmetric matrix

\[
A = (J + J^{n-3} + J^{n-1}) + (J + J^{n-3} + J^{n-1})^t
\]

with vanishing diagonal and \( v(A) = 2n + 6 \). The matrix \( A \) is odd-sized fully indecomposable \((0,1)\) symmetric matrices with vanishing diagonal.

**Example**

For \( n = 5 \)

\[
\begin{pmatrix}
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0
\end{pmatrix}
\]
Main blocks

Definition
Let $Q_k \in M_k(0, 1)$ denote the following matrix:

$$q_{ij} = \begin{cases} 
1, & \text{if } i + j = k \text{ or } i + j = k + 1; \\
0, & \text{otherwise.}
\end{cases}$$

Example

$$Q_1 = (1) \quad Q_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad Q_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad Q_4 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
Non-convetible fully indecomposable (0,1)-matrices with 2n + 3 ones

**Theorem**

Let $A \in M_n(0,1)$, $\nu(A) = 2n + 3$ and $A$ is non-convetible fully indecomposable (0,1)-matrix. Then there is matrix $K \in M_3(\mathbb{Z})$ with non-negative elements and sum of elements is equal to $n - 3$ such that $A$ is permutationally equivalent to matrix

$$
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{11} & O & O & Q_{k_{11}} & \cdots \\
a_{12} & O & Q_{k_{21}} & O & \cdots \\
a_{13} & Q_{k_{31}} & O & O & \cdots \\
b_1 & B_1 & O & O & \cdots \\
\end{pmatrix}
$$
Bipartite graphs

Definition
Graph $G = (V, E)$ is bipartied graph if $V = M \cup N$, $M \cap N = \emptyset$ and edge $e \in E$ is of the form $e = (n_i, m_j)$, where $n_i \in N$ and $m_j \in M$.

Definition
Matrix $A \in M_{n,m}(0,1)$ is adjacency matrix of bipartied graph $G = (V, E)$ if $a_{ij} = 1$ if and only if $e = (n_i, m_j) \in E$. 
Graph description

Graph $K_{3,3}$

$k = k_1$
Thank you!

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